Entry, Exit, and Strategic Entry

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Topics in Economic Theory

Entering a Market

A firm can enter a market by paying a cost that cannot be recovered.

Demand may improve, but entering now gives up the opportunity to wait.

When is the market attractive enough to enter?

Entry need not be permanent: the firm may leave and later return.

Should it leave as soon as operating profits become negative?

Access to the market may depend on outperforming other firms.

Submit a better project, or enter before a rival takes the opportunity.

How does competition change the incentive to wait?

The Tools

State X, flow payoff f, scrap value v, discount rate r:

$$V(x) = \sup_{\tau \in \mathbb{T}} \mathbb{E}^x \left[\int_0^\tau e^{-rt} f(X_t)\, dt + e^{-r\tau} v(X_\tau) \right].$$

Itô's lemma and the generator: how does a function of the state evolve?

DPP and HJB: in the continuation region,

$$(\mathcal{L} - r)V + f = 0 \quad \text{on } C.$$

Value matching: $v - V = 0$ on D.

Smooth fit: $v' - V' = 0$ on ∂C.

(Smooth fit may fail in many applications; care needs to be exercised here.)

Value matching and smooth fit: identify the stopping boundaries.

Optional sampling: calculate hitting probabilities and discounted payoffs.

Notation and Economic Interpretation

X_t: current state; x: its current value; τ: time of entry or exit.

V: value before stopping; v: scrap value upon stopping.

I: entry cost; E: exit cost; w: operating cost per unit of time.

μ: drift; σ: volatility; r: discount rate.

The economic meaning of the state will be explicit in each application.

- A capitalised value of entering the market.
- A flow revenue from operating in the market.
- A privately observed project value submitted to an entry contest.

n always denotes the *total* number of contestants.

Overview

1. Entry and the Value of Waiting
2. Entry, Exit, and Hysteresis
3. Entry Contests: Preemption
4. Entry Contests: Competition and Risk Taking
5. Concluding Remarks

Overview

1. Entry and the Value of Waiting
   - The Entry Problem
   - The Shape of the Entry Region
   - Deriving the Entry Threshold
   - Comparative Statics
   - Experimental Evidence

2. Entry, Exit, and Hysteresis

3. Entry Contests: Preemption

4. Entry Contests: Competition and Risk Taking

5. Concluding Remarks

Irreversible Entry

McDonald & Siegel (1986 QJE); Dixit & Pindyck (1994, ch. 5)

A risk-neutral firm can pay $I > 0$ to enter a market worth X_t.

- X_t is the present value of the profits obtained upon entry.
- The firm can wait indefinitely or never enter.
- For now, entry is irreversible.

Market value follows geometric Brownian motion:

$$dX_t = \mu X_t \, dt + \sigma X_t \, dB_t, \qquad \sigma > 0, \quad r > \max\{\mu, 0\}.$$

Positive net present value makes entry profitable. Does it make entry optimal?

Waiting preserves the ability to avoid entry after bad news.

Other Interpretations of Entry

Investment: pay a sunk cost to install a productive asset.

Taking a job: accept an opportunity whose capitalised benefit is X_t, paying a switching or relocation cost I.

Selling an asset: receive a stochastic sale price X_t, net of a transaction cost I.

Same stopping problem applies when waiting generates no other payoff and opportunity remains available.

Dividends before sale, unemployment benefits, or an expiring job offer change flow payoff or effective discount rate.

Entry interpretation as background application throughout.

The State Process and Its Generator

Itô's lemma. $\mathrm{d}f(X_t) = \left(f_t' + \mu_t f_x' + \frac{1}{2}\sigma_t^2 f_{xx}''\right)\mathrm{d}t + \sigma_t f_x' \,\mathrm{d}B_t$.

Applied to $\ln X_t$, where $\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}B_t$, gives

$$\mathrm{d}\ln X_t = \left(\mu - \sigma^2/2\right)\mathrm{d}t + \sigma\,\mathrm{d}B_t \iff \ln X_t = \ln x + \left(\mu - \sigma^2/2\right)t + \sigma B_t$$

$$\iff X_t = x\exp\left((\mu - \sigma^2/2)t + \sigma B_t\right) =: xZ_t.$$

Same Brownian path defines $X_t^x = xZ_t$ for every initial value.

Expected market value grows at rate μ: $\mathbb{E}^x X_t = xe^{\mu t}$.

(Take expectations; Z_t is log-normal.)

Infinitesimal generator

$$\mathcal{L}\phi(x) = \mu x\phi'(x) + \frac{1}{2}\sigma^2 x^2 \phi''(x).$$

Z_t is submartingale, but, since $r > \mu$, $e^{-rt}Z_t$ is a nonnegative supermartingale.

Waiting *reduces* expected *discounted* market value, but also postpones entry cost.

The Option to Enter

Entering immediately yields $x - I$; not entering yields zero.

We can write scrap value and option value as

$$v(x) = (x - I)^+, \qquad V(x) = \sup_{\tau \in \mathbb{T}} \mathbb{E}^x[e^{-r\tau} v(X_\tau)].$$

Terminal payoff is zero when $\tau = \infty$.

Equivalent to choosing an entry time, receiving $X_\tau - I$, and allowing never to enter.

For $x > 0$, fixed positive waiting time gives positive probability of $X_t > I$.

Hence $V(x) > 0$: at $0 < x \leq I$, neither entry nor permanent abandonment is optimal.

The firm waits for sufficiently favourable market conditions.

Properties of the Value Function

Proposition (Monotonicity, Convexity, and Lipschitz Bound)

V is increasing and convex. For $y > x > 0$,

$$0 \leq V(y) - V(x) \leq y - x, \qquad (x - I)^+ \leq V(x) \leq x.$$

Another pound of market value raises option value by at most one pound.

Firm receives extra value only if it eventually enters, and at discounted date.

Convexity captures ability to keep upside and avoid entry after sufficiently bad news.

Proof idea: compare same stopping rule on same Brownian path for two initial values.

Properties of the Value Function

Proof

Lipschitz Bound. For every Brownian stopping time τ, path by path,

$$0 \leq (yZ_\tau - I)^+ - (xZ_\tau - I)^+ \leq (y - x)Z_\tau.$$

Optional sampling gives $\mathbb{E}[e^{-r\tau}Z_\tau \mathbf{1}_{\{\tau<\infty\}}] \leq Z_0 = 1$. Consequently, for any $\tau \in \mathbb{T}$,

$$\mathbb{E}[e^{-r\tau}(yZ_\tau - I)^+\mathbf{1}_{\{\tau<\infty\}}] - \mathbb{E}[e^{-r\tau}(xZ_\tau - I)^+\mathbf{1}_{\{\tau<\infty\}}] \leq (y - x)\mathbb{E}[e^{-r\tau}Z_\tau \mathbf{1}_{\{\tau<\infty\}}] \leq (y - x)$$

Rearranging:

$$\begin{aligned}
\implies \mathbb{E}[e^{-r\tau}(yZ_\tau - I)^+\mathbf{1}_{\{\tau<\infty\}}] &\leq \mathbb{E}[e^{-r\tau}(xZ_\tau - I)^+\mathbf{1}_{\{\tau<\infty\}}] + (y - x) \\
&\leq \sup_\tau \mathbb{E}[e^{-r\tau}(xZ_\tau - I)^+\mathbf{1}_{\{\tau<\infty\}}] + (y - x) = V(x) + (y - x)
\end{aligned}$$

$$\implies \sup_\tau \mathbb{E}[e^{-r\tau}(yZ_\tau - I)^+\mathbf{1}_{\{\tau<\infty\}}] = V(y) \leq V(x) + (y - x).$$

Monotonicity and Convexity: Since $Z_\tau > 0$, $(xZ_\tau - I)^+$ is monotone and convex in x. Hence, $\mathbb{E}[(xZ_\tau - I)^+]$ also monotone and convex in x, and so is $V(x) = \sup_\tau \mathbb{E}[(xZ_\tau - I)^+]$.

One-Sided Threshold

Proposition (Entry Region)

On $[I, \infty)$, option premium $V(x) - (x - I)$ is nonnegative and nonincreasing. If entry is optimal at x, it is optimal $\forall y > x$.

Proof

For $y > x \geq I$, Lipschitz bound gives

$$V(y) - v(y) = V(y) - (y - I) \leq V(x) - (x - I) = V(x) - v(x).$$

Since $V \geq v$, once premium is zero ($V(x) = v(x)$) and entry is optimal, it stays zero and entry remains optimal.

Immediate entry gains one pound from another pound of market value;
waiting gains at most one pound.

Stationary Markov problem has an upper entry region, if nonempty.

We now establish threshold x^* is finite $I < x^* < \infty$.

Differential Characterisation of the Value Function

HJB: Before entry there is no flow payoff:

$$-rV(x) + \mu x V'(x) + \frac{1}{2}\sigma^2 x^2 V''(x) = 0, \qquad 0 < x < x^*.$$

Boundary conditions: $V(0) = 0$ and $V(x^*) = x^* - I$.

Smooth fit: $V'(x^*) = 1$.

Value matching makes firm indifferent between entry and waiting at the boundary.

Smooth fit removes a first-order gain from moving the boundary slightly.

Payoff kink is at I, strictly below entry boundary: scrap value is smooth at entry.

Solving for the Value Function

If we write $z := \ln x$ and $U(z) := V(e^z)$, then $U'(z) = xV'(x)$ and $U''(z) = x^2V''(x)$.

Change of variables:

$$-rU(x) + \mu U'(x) + \frac{1}{2}\sigma^2 U''(z) = 0, \qquad z < z^*.$$

Solution to ODE:

$$U(z) = K_+ e^{z\beta_+} + K_- e^{z\beta_-} \iff V(x) = K_+ x^{\beta_+} + K_- x^{\beta_-},$$

where $\beta_\pm = \frac{-(\mu-\sigma^2/2)\pm\sqrt{(\mu-\sigma^2/2)^2+2\sigma^2 r}}{\sigma^2}$ and $\beta_- < 0 < \beta_+$.

For any $K_- \neq 0$, $K_+x^{\beta_+} + K_-x^{\beta_-}$ diverges as $x \to 0$, which contradicts $0 \leq V(x) \leq x$.

Conclude $K_- = 0$.

$\implies V(x) = K_+x^{\beta_+}, \;\; x < x^*.$

Entry region cannot be empty: otherwise this form holds for all $x > 0$, contradicting $0 < V(x) \leq x$.

Solving for the Entry Boundary

Substitute $V(x) = K_+ x^{\beta_+}$ into value matching and smooth fit:

$$K_+(x^*)^{\beta_+} = x^* - I, \qquad K_+\beta_+(x^*)^{\beta_+ - 1} = 1.$$

Dividing the second equation by the first yields

$$\frac{\beta_+}{x^*} = \frac{1}{x^* - I} \qquad \Longrightarrow \qquad x^* = \frac{\beta_+}{\beta_+ - 1} I.$$

Then $K_+ = (x^* - I)/(x^*)^{\beta}$.

Entry threshold covers both sunk cost and value of giving up option to wait.

Higher boundary offers larger entry payoff but makes reaching it less likely and more heavily discounted.

Optimal Entry

Proposition (Entry Time and Value)

Let $\beta := \frac{-(\mu-\sigma^2/2)+\sqrt{(\mu-\sigma^2/2)^2+2\sigma^2 r}}{\sigma^2}$ and $x^* = \frac{\beta}{\beta-1} I$.
The optimal stopping time and value function are given by

$$\tau^* = \inf\{t \geq 0 : X_t \geq x^*\}, \qquad V(x) = \begin{cases} (x^* - I)(x/x^*)^\beta, & 0 \leq x < x^*, \\ x - I, & x \geq x^*. \end{cases}$$

The required proportional entry premium is $(x^* - I)/I = 1/(\beta - 1)$.

Entry can optimally never occur on paths that fail to reach x^*.

Profitable market can be worth waiting to enter.

Entry as a Choice of Threshold

Claim: $e^{-rt}X_t^\beta$ **is martingale.**

Proof: By Itô's lemma:

$$\mathrm{d}(e^{-rt}X_t^\beta) = e^{-rt}X_t^\beta\left[(\frac{1}{2}\sigma^2\beta(\beta-1)+\mu\beta-r)\,\mathrm{d}t+\beta\sigma\,\mathrm{d}B_t\right].$$

Since $\beta : (\frac{1}{2}\sigma^2\beta(\beta-1)+\mu\beta-r) = 0$, then

$$\mathrm{d}(e^{-rt}X_t^\beta) = e^{-rt}X_t^\beta\beta\sigma\,\mathrm{d}B_t$$

and $\mathbb{E}[\mathrm{d}(e^{-rt}X_t^\beta) \mid \mathcal{F}_t] = 0$ and $\mathbb{E}\int_0^T(e^{-rt}X_t^\beta)^2\,\mathrm{d}t < \infty$.

Optional stopping: As $e^{-rt\wedge\tau_a}X_{t\wedge\tau_a}^\beta \in [0,a]$ a.s.,

$$\mathbb{E}[e^{-r\tau_a}X_{\tau_a}^\beta] = x^\beta = a^\beta\mathbb{E}[e^{-r\tau_a}1_{\{\tau_a<\infty\}}]$$

$$\iff \mathbb{E}[e^{-r\tau_a}1_{\{\tau_a<\infty\}}] = (x/a)^\beta.$$

Entry at $a \geq I$ yields

$$(a-I)(x/a)^\beta.$$

Entry as a Choice of Threshold

Entry at $a \geq I$ yields

$$(a - I)(x/a)^{\beta}.$$

Its derivative with respect to a has the sign of

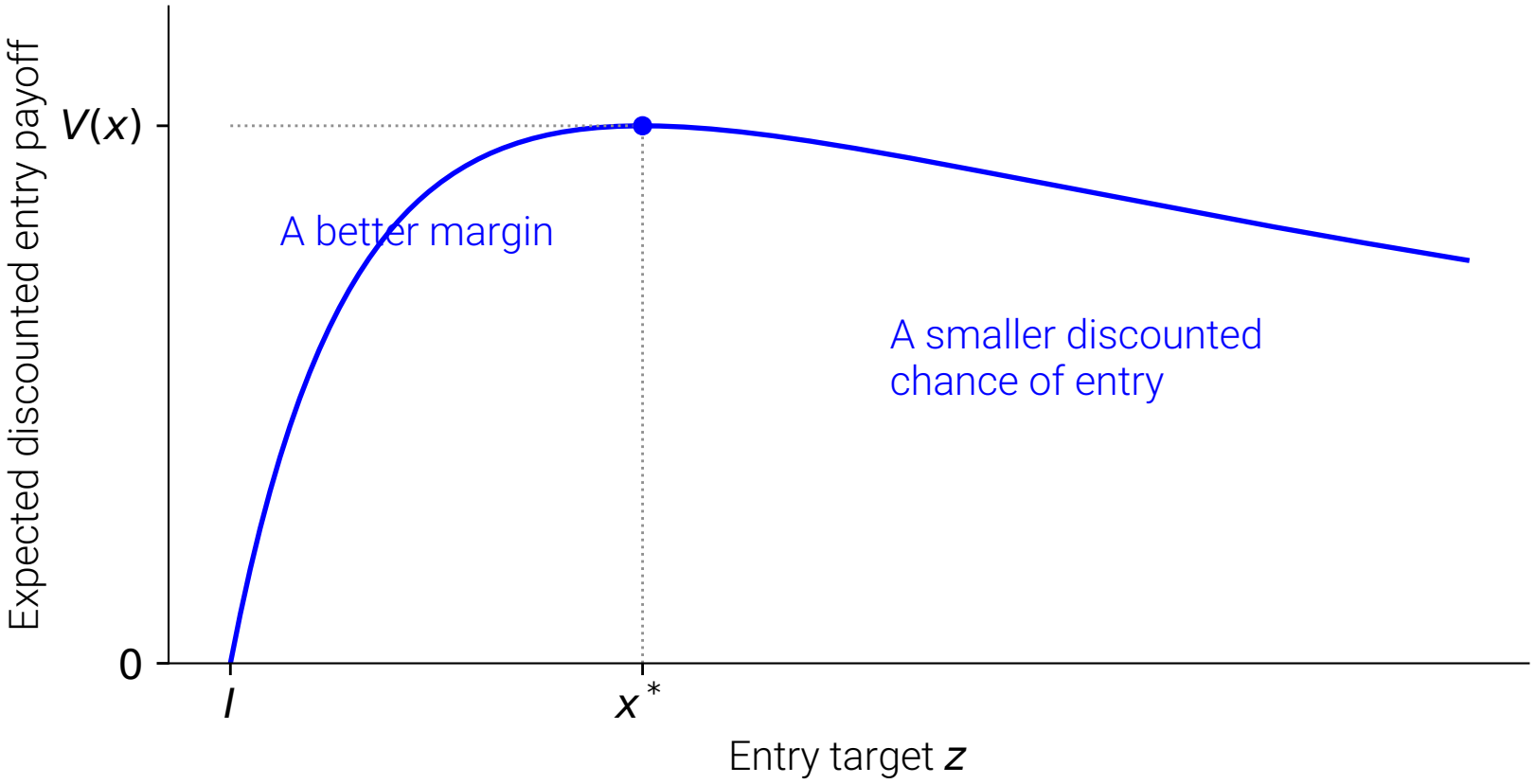
$$a - \beta(a - I).$$

It is positive below x^* and negative above x^*.

Same boundary as value matching and smooth fit.

Power term = discounted probability of entry.

Choosing the Entry Target

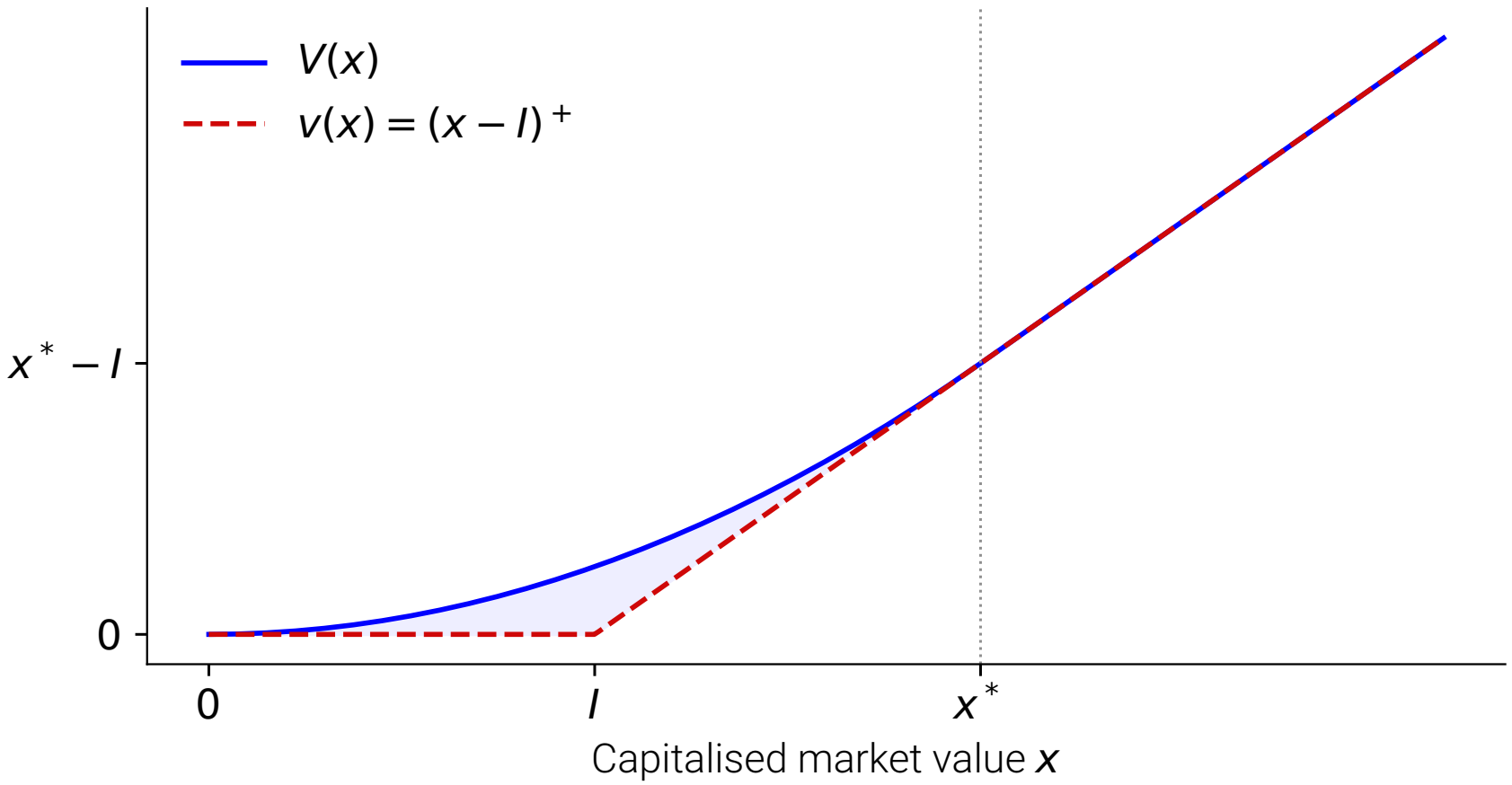


Raising the target increases the margin conditional on entry.

Beyond x^*, the loss in discounted hitting probability dominates.

The optimal target maximises $(a - l)(x/a)^{\beta}$.

Positive NPV Is Not Enough



At $I < x < x^*$, entry profitable but waiting is better.

$V - v$ measures option value of entry decision.

At x^*, value curve tangent to entry payoff (smooth fit).

Entry: Comparative Statics

Proposition (Comparative Statics)

Holding other primitives fixed,

$$\frac{\partial x^*}{\partial I} > 0, \qquad \frac{\partial x^*}{\partial \sigma} > 0, \qquad \frac{\partial x^*}{\partial \mu} > 0, \qquad \frac{\partial x^*}{\partial r} < 0.$$

Threshold proportional to I. Proportional entry premium independent of I.

Higher sunk cost: require more valuable market.

More uncertainty: flexibility more valuable.

Faster expected appreciation: waiting becomes more attractive.

More discounting: future entry opportunities become less attractive.

Note: higher threshold does not imply longer expected entry time when the state process also changes.

Moreover, note that $\mu < \sigma^2/2 \iff \mathbb{E}[\tau_a] = \infty \; \forall a > x$.

Market Value versus Current Revenue

Here x already capitalises all profits obtained upon entry.

If instead current revenue is y, follows GBM, and is earned forever after entry, then

$$x = y/(r - \mu) \implies y^* = (r - \mu)\frac{\beta}{\beta - 1}I$$

Increasing r lowers x^* but raises y^*.

Increasing μ raises x^* but lowers y^*.

Increase in value of waiting and an increase in value of operating business are different effects.

Next, entry and exit will use current revenue as the state.

But first...

Learning to Wait: An Experiment

Oprea, Friedman, & Anderson (2009 REStud)

Participants observe changing market value and choose when to pay an entry cost.

Entry earns $X_\tau - I$; no entry before expiration earns zero.

The experiment makes the cost, the state process, and the decision observable.

Prediction: entry requires a premium

$$\omega^* := \frac{x^* - I}{I} = \frac{1}{\beta - 1}.$$

Does behaviour track predicted premium?

Implementing the Entry Problem

Additive shocks of size $\sigma\sqrt{\Delta}$ converge to Brownian motion.

Here shocks *proportional* to current value:

$$X_{t+\Delta} = \begin{cases} (1+h)X_t, & \text{with probability } p, \\ (1-h)X_t, & \text{with probability } 1-p. \end{cases}$$

Conditional on $X_t = x$,

$$\mathbb{E}[\Delta X_t \mid x] = (2p-1)hx, \qquad \mathrm{Var}(\Delta X_t \mid x) = 4p(1-p)h^2x^2.$$

Discretisation approximates $\mu x\Delta$ and $\sigma^2 x^2 \Delta$:

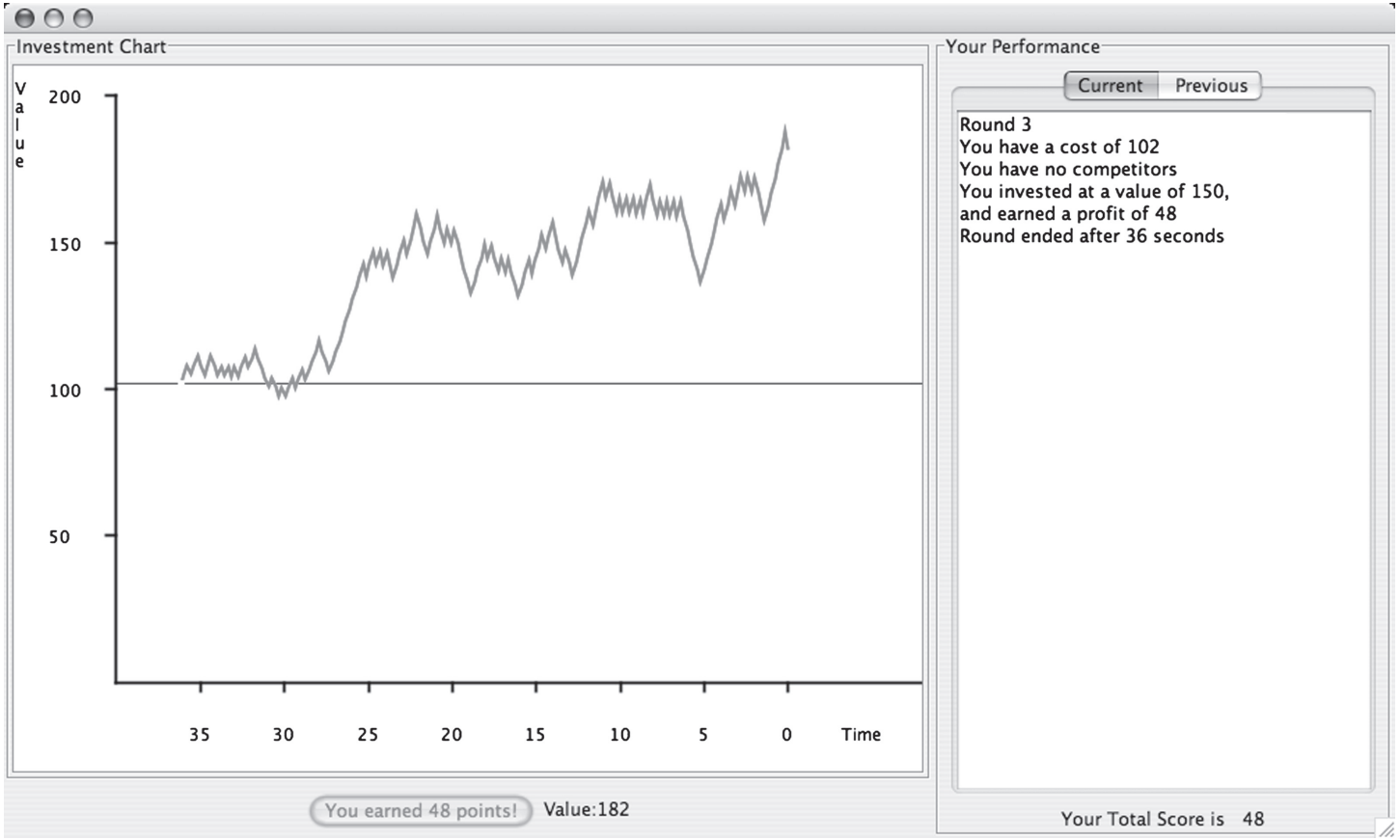
$$\mu \approx \frac{(2p-1)h}{\Delta}, \qquad \sigma^2 \approx \frac{4p(1-p)h^2}{\Delta}.$$

Random-walk limit is GBM. Step size h and uptick probability p control the value process.

Random termination implements discounting and controls cost of waiting.

More variable percentage changes implement higher volatility of market value.

Learning to Wait: The Experimental Interface



Oprea, Friedman, & Anderson (2009 REStud), Figure 2. End-of-period screen.

Moving line shows market value; horizontal line shows participant's entry cost.

Right panel reports decision and earnings. Path continues after entry, providing feedback on timing decision.

Experimental Design

69 participants; 80 paid rounds in a treatment.

Entry cost redrawn each period; value process starts at cost.

Values update five times per second.

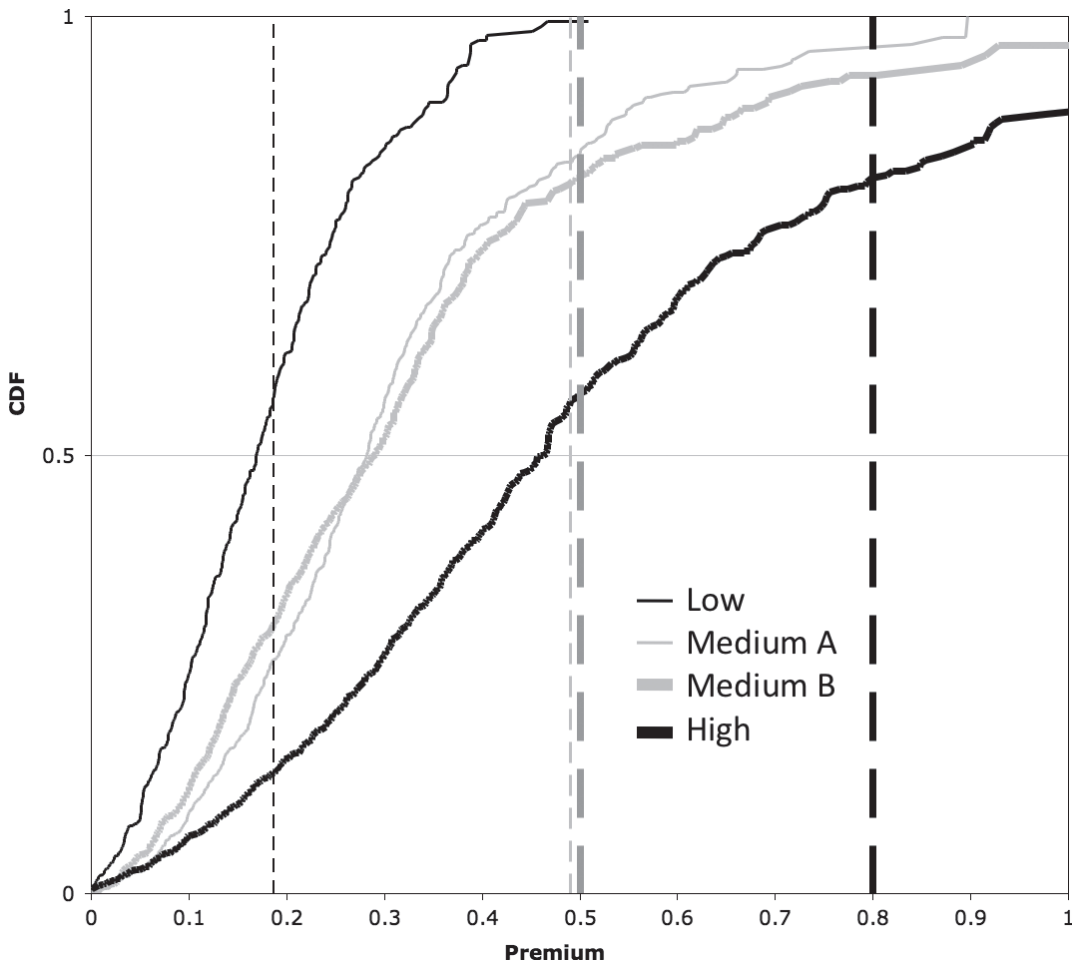
Four treatments test economic trade-offs.

Vary random walk and expiration parameters:

$$\omega^*_{\text{Low}} < \omega^*_{\text{Medium A}} \simeq \omega^*_{\text{Medium B}} < \omega^*_{\text{High}}.$$

Two medium treatments reach similar predicted premia through different parameter combinations.

Entry Premia: The Distribution of Decisions



Oprea, Friedman, & Anderson (2009 REStud), Figure 4. Premium is $(X_\tau - I)/I$.

Censoring-adjusted cdfs of premia.

Vertical lines: theoretical premia.

Low, Medium, and High treatments ordered as predicted.

In Medium and High, much of distribution lies below theoretical target.

Observed Entry Premia

	Low	Medium A	Medium B	High
Predicted premium	.179	.490	.499	.804
Mean estimated premium	.182	.310	.328	.575

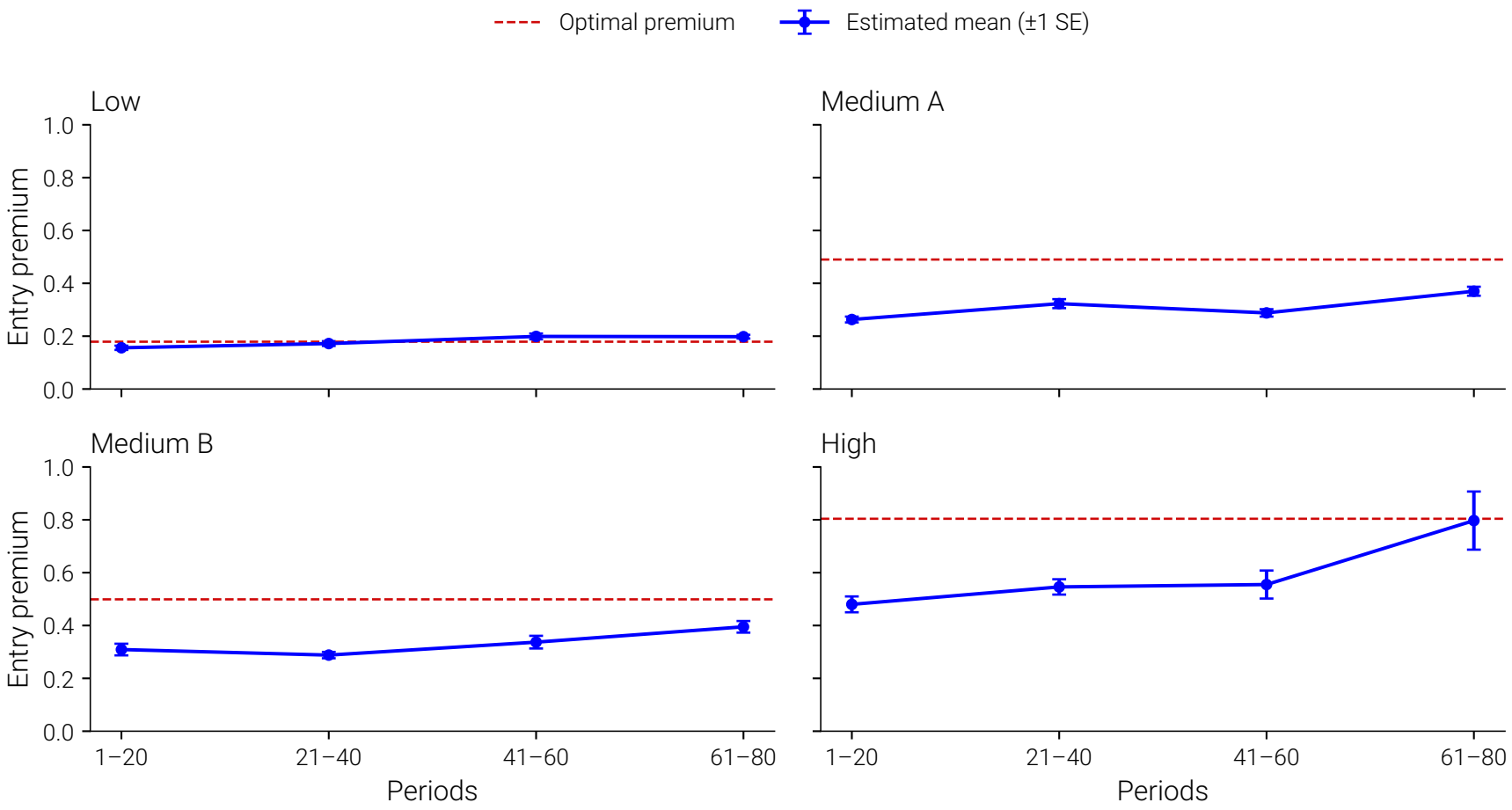
Ordering across treatments matches theory.

Low-treatment behaviour is close to predicted premium.

In Medium and High treatments, entry at lower values than predicted.

People understand option value of waiting, but tend to undervalue it.

Learning Across Repeated Entry Decisions



Oprea, Friedman, & Anderson (2009), Table 4. Pooled product-limit means; bars show ±1 standard error.

Dashed lines are optimal premia. Later decisions generally closer to theory predictions.

Overview

1. Entry and the Value of Waiting

2. Entry, Exit, and Hysteresis
   - Entry and Exit
   - Solving for Entry and Exit
   - Comparative Statics

3. Entry Contests: Preemption

4. Entry Contests: Competition and Risk Taking

5. Concluding Remarks

Entering and Leaving a Market

Dixit (1989 JPE)

Active firm earns revenue X_t and pays operating cost $w > 0$ per unit of time:

$$\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}B_t.$$

Entry costs $I > 0$; exit costs $E \geq 0$; re-entry possible at cost I.

Idle firm receives zero flow payoff.

Assume $\sigma > 0$, $r > \max\{\mu, 0\}$, and $w > rE$.

$V_0(x)$: value when idle. $V_1(x)$: value when active.

Relevant state: $(x, \text{current regime})$.

Model can describe hiring and firing, or entering and leaving an export market, buying and selling an asset.

Entry and Exit as Options Linked Options: Scrap Values

On entry, idle firm pays I and becomes an active firm.

It receives an operating business with *right to exit and re-enter*.

$$v_0(x) = V_1(x) - I.$$

On exit, active firm pays E and becomes an idle firm.

Retains right to enter again when market conditions improve.

$$v_1(x) = V_0(x) - E.$$

DPP: Stopping payoff includes optimally chosen continuation after current decision.

Entry and Exit as Options Linked Options: Optimal Switching Problem

Idle firm:

$$V_0(x) = \sup_{\tau} \mathbb{E}^x[e^{-r\tau}(V_1(X_\tau) - I)].$$

Active firm:

$$V_1(x) = \sup_{\tau} \mathbb{E}^x \left[\int_0^\tau e^{-rt}(X_t - w)\, dt + e^{-r\tau}(V_0(X_\tau) - E) \right].$$

Never entering earns zero; never exiting earns perpetual operating profits.

Switching sequences cannot accumulate in finite time.

Coupled stopping problems: value of entry depends on value of exit, and value of exit depends on value of re-entry.

Entry and Exit as Options Linked Options: Variational Inequalities

HJBs and Boundary Conditions

$$\max\{V_1 - I - V_0,\ (\mathcal{L} - r)V_0\} = 0,$$
$$\max\{V_0 - E - V_1,\ (\mathcal{L} - r)V_1 + x - w\} = 0.$$

Solution has lower exit threshold x_E and upper entry threshold x_I.

Idle: enter when $X_t \geq x_I$.

Active: exit when $X_t \leq x_E$.

Positive entry-exit cost $I + E$ separates two decisions.

Inside (x_E, x_I), continue in current regime.

Firms may operate at a loss:

Active firm with $x < w$ is losing money. But revenue may recover, exit costs money, and returning requires another entry cost.

Idle firm at same revenue can avoid losses without paying to exit.

Solving for the Switching Thresholds

Idle firm receives no flow payoff, so, for similar reasons as before:

$$V_0(x) = K_+ x^{\beta_+}, \qquad x < x_I.$$

Active firm receives $x - w$, giving

$$V_1(x) = \frac{x}{r - \mu} - \frac{w}{r} + K_- x^{\beta_-}, \qquad x > x_E.$$

First two terms: value of remaining active forever.

Final term: option value to exit and subsequently re-enter.

Positive-power term (β_+) drops out: at infinity, value grows at most linearly.

Solving for the Switching Thresholds

The unknowns are K_+, K_-, x_E, and x_I.

Value matching and smooth fit, now at both switching boundaries:

$$V_0(x_I) = V_1(x_I) - I, \qquad V_0'(x_I) = V_1'(x_I),$$
$$V_1(x_E) = V_0(x_E) - E, \qquad V_1'(x_E) = V_0'(x_E).$$

At entry, paying I and changing regime leaves value unchanged.

At exit, paying E and changing regime leaves value unchanged.

Fixed switching cost has zero derivative, so smooth fit is plausible.

Kink would create first-order gain from shifting switching point.

Solving for the Switching Thresholds

Define the value difference

$$\Delta(x) := V_1(x) - V_0(x).$$

In the overlap of the continuation regions,

$$\Delta(x) = \frac{x}{r-\mu} - \frac{w}{r} + K_- x^{\beta_-} - K_+ x^{\beta_+}.$$

The four boundary conditions become

$$\Delta(x_E) = -E, \quad \Delta'(x_E) = 0, \qquad \Delta(x_I) = I, \quad \Delta'(x_I) = 0.$$

Entry worthwhile when advantage of being active reaches its cost.

Exit worthwhile when disadvantage reaches the cost of leaving.

Between these points, $-E < \Delta(x) < I$ and $\Delta'(x) > 0$.

Solving for the Switching Thresholds

For any candidate $0 < x_E < x_I$, two slope conditions are linear in K_+, K_-:

$$K_+\beta_+ x_j^{\beta_+ - 1} - K_-\beta_- x_j^{\beta_- - 1} = \frac{1}{r - \mu}, \qquad j \in \{L, H\}.$$

Solve two-by-two system for $K_+(x_E, x_I)$ and $K_-(x_E, x_I)$.

Substitute into two value conditions:

$$\Delta(x_E; x_E, x_I) + E = 0, \qquad \Delta(x_I; x_E, x_I) - I = 0.$$

Equations determine two triggers; generally no elementary closed form.

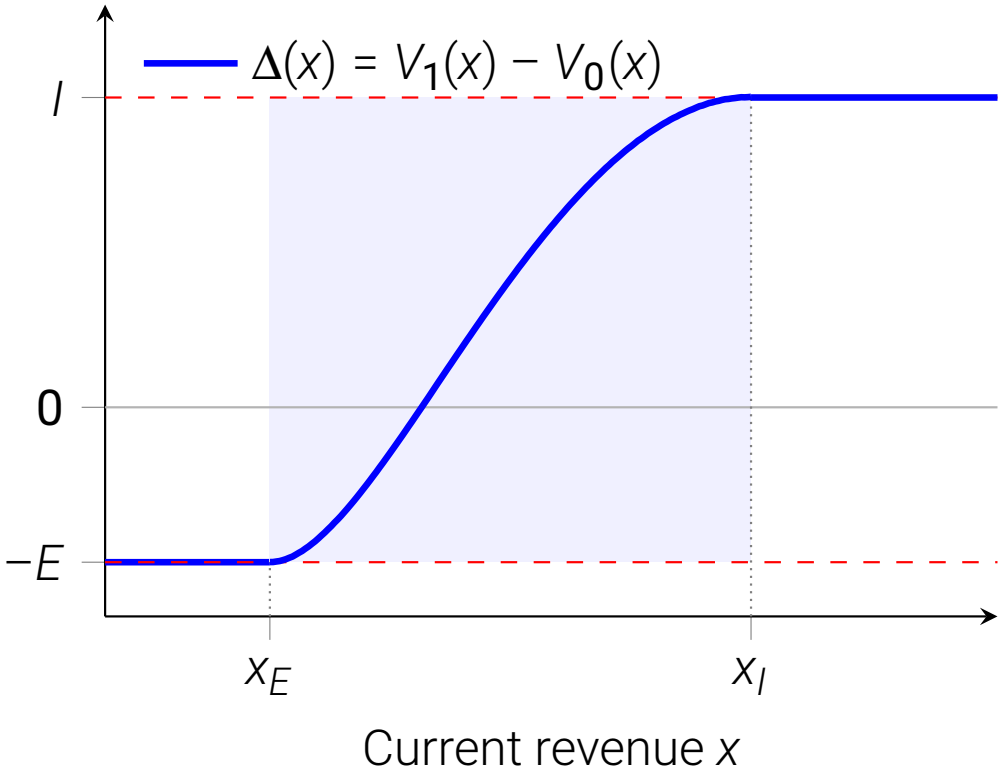
Reduces two coupled stopping problems to two boundary equations.

Proposition (Entry and Exit)

The firm enters at x_I when idle and exits at x_E when active, with $0 < x_E < x_I$, where x_E, x_I solve:

$$\Delta(x_E) = -E, \quad \Delta'(x_E) = 0, \qquad \Delta(x_I) = I, \quad \Delta'(x_I) = 0.$$

Switching Thresholds



Enter when advantage of activity reaches I; exit when it reaches $-E$.

Smooth fit makes both contacts horizontal.

Between thresholds, neither switch pays for itself.

Quick question: *How do we even know that Δ is increasing in (x_E, x_I)?*

Switching Thresholds

On (x_E, x_I), $(\mathcal{L} - r)\Delta = (\mathcal{L} - r)(V_1 - V_0)$, hence Δ satisfies:

$$-r\Delta(x) + \mu x\Delta'(x) + \frac{1}{2}\sigma^2 x^2 \Delta''(x) = (\mathcal{L} - r)\Delta(x) = (\mathcal{L} - r)(V_1 - V_0)(x) = w - x.$$

Here's a useful trick: Define $q := \Delta'$ and differentiate HJB above wrt x; q satisfies

$$-(r - \mu)q + (\mu + \sigma^2)xq' + \frac{1}{2}\sigma^2 x^2 q'' = -1, \qquad q(x_E) = q(x_I) = 0.$$

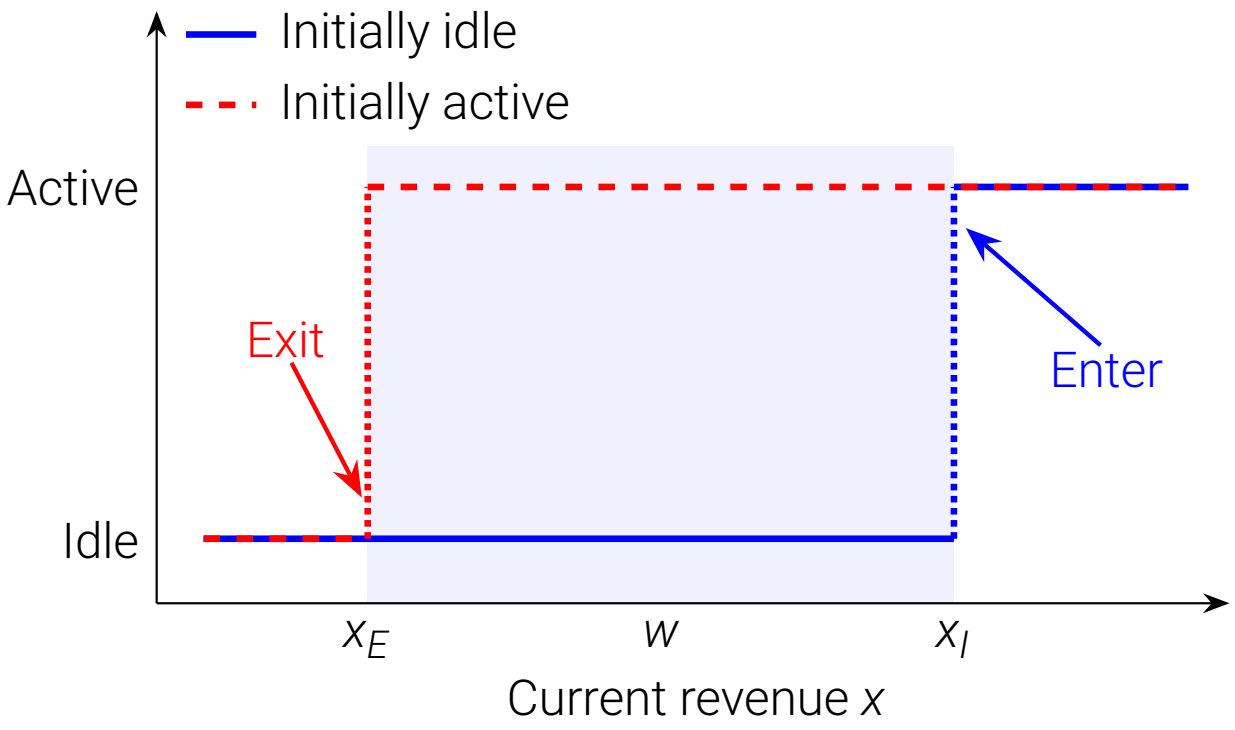
Interpret LHS using GBM with drift $\mu + \sigma^2$ and discount rate $r - \mu > 0$.

Dynkin's formula gives

$$q(x) = \mathbb{E}^x \int_0^{\widetilde{\tau}_{E,I}} e^{-(r-\mu)t}\, dt > 0.$$

Hence, Δ strictly increasing between (x_E, x_I)!

Hysteresis/Path Dependency



Rising revenue triggers entry at x_I; falling revenue triggers exit at x_E.

Within the interval, activity depends on the firm's past entry decision.

Temporary improvement can cause entry that persists after conditions return to their previous level.

Entry and Exit: Comparative Statics

Proposition (Comparative Statics)

Increasing σ^2, I, or E expands the (x_E, x_I).
Increasing w or decreasing μ, shifts (x_E, x_I) up.

Volatility and switching costs widens the hysteresis region.

Higher volatility: (1) more likely to switch at fixed thresholds, (2) greater option value of waiting; expand (x_E, x_I).

Higher operating costs make activity less attractive at both margins.

Faster revenue growth encourages entry and discourages exit.

How to prove it?

Differentiating the Switching Boundaries

Boundary conditions implicitly define x_E, and x_I.

$$\Delta(x_E) = -E, \quad \Delta'(x_E) = 0, \qquad \Delta(x_I) = I, \quad \Delta'(x_I) = 0.$$

For any primitive θ, write $D_\theta := \partial_\theta \Delta$.

Then, differentiate smooth fit condition and use implicit function theorem:

$$\frac{\partial x_j}{\partial \theta} = -\frac{D'_\theta(x_j)}{\Delta''(x_j)}.$$

Entry and exit costs

$$(\mathcal{L} - r)D_I = (\mathcal{L} - r)D_E = 0,$$

$$D_I(x_E) = 0, \quad D_I(x_I) = 1, \qquad D_E(x_E) = -1, \quad D_E(x_I) = 0.$$

Use Dynkin's formula: $D_I(x) := \mathbb{E}^x \int_0^{\tau_{E,I}} e^{-rt} \mathbf{1}_{\{X_{\tau_{E,I}} = x_I\}} \, dt$.

D_I is the discounted probability of hitting x_I first.

D_E is minus the discounted probability of hitting x_E first.

Both increase with x: positive boundary slopes lower x_E and raise x_I.

Differentiating the Switching Boundaries

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Then, differentiate smooth fit condition and use implicit function theorem:

$$\frac{\partial x_j}{\partial \theta} = -\frac{D'_\theta(x_j)}{\Delta''(x_j)}.$$

For other primitives: different derivations, but **same proof principles:**

(1) Define D_θ. Get differential characterisation of D_θ by differentiating HJB for Δ and boundary/smooth-fit conditions.

(2) Use Dynkin's formula. Sign D'_θ.

Try it yourself with w and μ. (For r and σ more involved.)

HJB very powerful to obtain comparative statics.

What Do Entry and Exit Reveal?

Market can be profitable and still inactive due to entry option value.

Continued operation at a loss need not indicate mistaken expectations or market power.

Current revenue is insufficient to predict activity; current regime matters: hysteresis/path-dependency.

Policy that changes entry costs also changes exit decisions of incumbents.

Persistent effects of temporary changes in profitability when they trigger entry or exit.

Next: stopping payoff itself will depend on competitors' decisions.

Overview

1. Entry and the Value of Waiting

2. Entry, Exit, and Hysteresis

3. Entry Contests: Preemption
   - Preemption Game
   - Characterising Equilibrium
   - Comparative Statics
   - Experimental Design and Evidence

4. Entry Contests: Competition and Risk Taking

5. Concluding Remarks

Preemption Game

Anderson, Friedman, & Oprea (2010 AER)

Market can sustain only one entrant. All firms observe market's capitalised value X_t.

First entrant obtains X_τ and pays its own entry cost I_i.

All other firms receive zero.

Each firm knows its own cost I_i, but not its rivals' costs.

Waiting offers higher potential profit margin and risk of being preempted.

How much of the monopoly option value to wait survives competition?

Preemption Game

Players: $n \geq 2$ firms. **Common public state:** $dX_t = \mu X_t \, dt + \sigma X_t \, dB_t$, $X_0 = x$.

Costs: $l_i \overset{\text{iid}}{\sim} H$, with continuous density $h > 0$ on $[\underline{l}, \bar{l}]$, where $0 < \underline{l} < \bar{l}$. Assume $x \leq \underline{l}$.

Discounting: $r > \max\{\mu, 0\}$, $\sigma > 0$.

Strategies: stopping times τ_i adapted to natural filtration.

Payoffs: $v(x; l_i) = x - l_i$ is stop first, otherwise zero; ties broken symmetrically.

The Monopoly Benchmark

With one firm, firm of cost l enters at

$$b_M(l) = \frac{\beta}{\beta - 1} l, \qquad \frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r = 0.$$

Trade-off: higher threshold $\uparrow$ profit conditional on entry but $\downarrow$ discounted prob.

With rivals, higher threshold also increases chance that another firm enters first.

Look for symmetric equilibrium threshold $b : [\underline{l}, \bar{l}] \to \mathbb{R}$, strictly increasing, differentiable.

- Lower-cost firms can afford to enter at lower market value.
- Strictly increasing $\implies$ ties have prob. zero.
- We'll assume continuity, but it could be derived.

A Firm's Expected Payoff

Suppose firm has cost l and rivals enter according to b.

Claim: Entering at $\hat{x} < l$ is suboptimal $\implies b(l) \geq l$.

Claim: Entering at $\hat{x} > b(\bar{l})$ is suboptimal.

Proof: It gets zero payoff wp1 and for $l < \bar{l}$ deviating to $b(\bar{l}) - \varepsilon$ is strictly profitable.

Claim: Entering at $\hat{x} < b(\underline{l})$ suboptimal only if $b(\underline{l}) \leq b_M(\underline{l})$.

Proof: Suppose otherwise: $b_M(\underline{l}) < b(\underline{l})$. Then, firm with l in neighbourhood of $\underline{l}$ attains monopoly payoff by deviating and entering at lower threshold $b_M(\underline{l})$.

A Firm's Expected Payoff

Suppose firm has cost l and rivals enter according to b.

Claim: Entering at $\hat{x} < l$ is suboptimal $\implies b(l) \geq l$.

Claim: Entering at $\hat{x} > b(\bar{l}) \geq \bar{l}$ is suboptimal.

Claim: Entering at $\hat{x} < b(\underline{l})$ suboptimal only if $b(\underline{l}) \leq b_M(\underline{l})$.

Only deviations left to consider are to $a = b(J) \in [b(\underline{l}), b(\bar{l})]$.

Now we deal with characterising b s.t. $b(l)$ is a best-response to b, $\forall l \in [\underline{l}, \bar{l}]$.

Claim: As $x \leq \underline{l} \leq b(\underline{l})$, expected payoff from imitating cost type J and entering at $b(J)$ is

$$u(J, l) := \underbrace{(b(J) - l)}_{\text{entry profit}} \underbrace{\left(\frac{x}{b(J)}\right)^{\beta}}_{\text{discounted hitting probability}} \underbrace{[1 - H(J)]^{n-1}}_{\text{all rivals have higher costs}} .$$

Proof: For monopolist, expected payoff from entry at $X_t \geq a$ is $(a - l)(x/a)^{\beta}$.

Claim then follows from independence of costs. ▷ Earlier derivations

A Firm's Expected Payoff

Suppose firm has cost l and rivals enter according to b.

Claim: Entering at $\hat{x} < l$ is suboptimal $\implies b(l) \geq l$.

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Let ι be s.t. $\iota(a) := b^{-1}(a)$, $\forall a \in (b(\underline{l}), b(\bar{l}))$; $\iota(a) := \underline{l}$ for $a \leq b(\underline{l})$; and $\iota(a) := \bar{l}$ for $a \geq b(\bar{l})$.

More generally, ex-ante expected payoff from entering at a is

$$(a - l)\left(\frac{x}{a}\right)^{\beta}[1 - H(\iota(a))]^{n-1}.$$

Dynamic Consistency?

But X_t is also informative about others' private costs!

Firm i's beliefs about expected payoffs are revised consistently.

Let z be highest market value observed so far. If no rival entered, threshold exceeds $\iota(z)$.

Is it better to revise target threshold?

Future target $a > z$ wins against all rivals with conditional probability

$$\left(\frac{1 - H(\iota(a))}{1 - H(\iota(z))}\right)^{n-1}.$$

Denominator does not depend on chosen a. Hence, if a maximises expected payoff, then it also maximises the expected payoff conditional on $\max_{s \leq t} X_t = z$, i.e., [expected payoff]/$(1 - H(\iota(z)))^{n-1}$.

Same threshold rule remains optimal after updating about surviving rivals.

Public history affects beliefs and expected payoff, without changing optimality of entry threshold strategy for surviving cost type.

Envelope Theorem

In equilibrium, type I prefers to imitate itself: $J = I$ maximises $u(J, I)$. Let

$$p(J) := \left(\frac{x}{b(J)}\right)^{\beta} [1 - H(J)]^{n-1}; \qquad u(J, I) := (b(J) - I)p(J); \qquad U(I) := u(I, I).$$

Envelope theorem: $U'(I) = -p(I)$. Note that $U(\bar{I}) = 0$.

Candidate Equilibrium Strategy in Integral Form: Since $(b(I) - I)p(I) = U(I) = \int_I^{\bar{I}} p(\widehat{I})\, \mathrm{d}\widehat{I}$, dividing by $p(I) > 0$,

$$b(I) = I + \int_I^{\bar{I}} \frac{p(\widehat{I})}{p(I)}\, \mathrm{d}\widehat{I} = I + \int_I^{\bar{I}} \left(\frac{b(I)}{b(\widehat{I})}\right)^{\beta} \left(\frac{1 - H(\widehat{I})}{1 - H(I)}\right)^{n-1} \mathrm{d}\widehat{I}.$$

and $b(\bar{I}) = \bar{I}$.

Candidate Equilibrium Strategy ODE

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Envelope theorem: $U'(I) = -p(I)$. $\quad U(\bar{I}) = 0$ and $b(\bar{I}) = \bar{I}$.

Differentiating $u(I, I)$ wrt I gives $(b'(I) - 1)p(I) + (b(I) - I)p(I)$.

Equating the two: $(b'(I) - 1)p(I) + (b(I) - I)p(I) = -p(I) \iff b'(I)p(I) + (b(I) - I)p'(I) = 0$.

Since $\ln p(I) = \beta \ln(x/b(I)) + (n - 1)\ln(1 - H(I))$, then

$$\frac{p'(I)}{p(I)} = -\beta\frac{b'(I)}{b(I)} + (n - 1)\frac{h(I)}{1 - H(I)}.$$

Plugging into $b'(I)p(I) + (b(I) - I)p'(I) = 0$ and rearranging we get

$$\frac{b'(I)}{b(I) - I} = \beta\frac{b'(I)}{b(I)} + (n - 1)\frac{h(I)}{1 - H(I)} = 0.$$

Candidate Equilibrium Strategy ODE

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and $b(\bar{l}) = \bar{l}$.

Candidate Equilibrium Strategy ODE:

$$\frac{b'(l)}{b(l) - l} = \beta \frac{b'(l)}{b(l)} + (n-1)\frac{h(l)}{1 - H(l)} = 0.$$

mg benefit of higher target = time cost + additional preemption risk.

Comments:

Without rivals ($n = 1$), last term disappears and we recover monopoly threshold.

Also cool relation: $\beta = 0$ corresponds exactly to procurement 1st-price auction with independent private values/costs.

Rearranging ODE:

$$b'(l) = (n-1)\frac{h(l)}{1 - H(l)} b(l) \frac{b(l) - l}{b(l) - \beta(b(l) - l)}.$$

Candidate Equilibrium Strategy ODE

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Rearranging ODE:

$$b'(l) = (n-1)\frac{h(l)}{1 - H(l)} b(l) \frac{b(l) - l}{b(l) - \beta(b(l) - l)}.$$

Candidate Equilibrium Strategy

Optimality:

Since $U(I) = (b(I) - I)p(I) = \int_I^{\bar{I}} p(\widehat{I})\, \mathrm{d}\widehat{I}$, imitating J yields
$u(J, I) = (b(J) - I)p(J) = U(J) + (J - I)p(J)$.

For $I < J$, $U(I) - u(J, I) = U(I) - U(J) - (J - I)p(J) = \int_I^J (p(\widehat{I}) - p(J))\, \mathrm{d}\widehat{I} \geq 0$.

For $I > J$, $U(I) - u(J, I) = U(I) - U(J) - (J - I)p(J) = \int_J^I (p(J) - p(\widehat{I}))\, \mathrm{d}\widehat{I} \geq 0$.

Monotonicity: from ODE $b' > 0$.

What about existence/uniqueness of (differentiable) b? It exists and is unique. ▷ Details

Equilibrium Entry Threshold

Proposition (Symmetric Bayesian Equilibrium)

There is a symmetric threshold equilibrium characterised by

$$b'(l) = (n-1)\frac{h(l)}{1-H(l)}\frac{b(l)[b(l)-l]}{b(l)-\beta[b(l)-l]}, \qquad l \in (\underline{l}, \bar{l}),$$

with $b(\bar{l}) = \bar{l}$. Moreover, it satisfies $l < b < b_M, b_{FPA}$, where b_M solves the same equation with $n = 1$ and b_{FPA} with $\beta = 0$

Equilibrium unique within class of symmetric threshold equilibria.

Firm enters when public value first reaches its cost-dependent threshold.

Highest-cost type cannot sustain positive profit margin against lower-cost rivals.

Monopoly bound b_M removes strategic preemption.

Auction bound b_{FPA} removes discounted cost of reaching higher target; vanishes at $\bar{l}$.

Both passage of time and competition constrain premium a firm can demand.

Preemption: Comparative Statics

Proposition (Comparative Statics)

For fixed cost distribution and interior cost type, the equilibrium entry thresholds:

Decreases in the number of contestants, n.

Decreases in β, i.e., decreases in discount rate r, increases in drift μ and volatility σ.

Increases in own cost l.

Cost, Entry Values, and Profit Margins

Since $b' > 0$, lowest-cost firm enters first whenever entry occurs.

$\implies$ selects most efficient entrant from available firms.

Higher entry value need not mean a higher profit margin:

$$\frac{\mathrm{d}}{\mathrm{d}l}[b(l) - l] = b'(l) - 1.$$

Sign depends on distribution. For uniform costs, b is concave and approaches the auction bound tangentially at $\bar{l}$.

Preemption in the Laboratory

Anderson, Friedman, & Oprea compare monopoly entry with entry in three-firm markets.

72 participants; two parameter treatments, with 36 participants in each.

Each session: 10 monopoly periods, 25 competition periods, 10 monopoly periods.

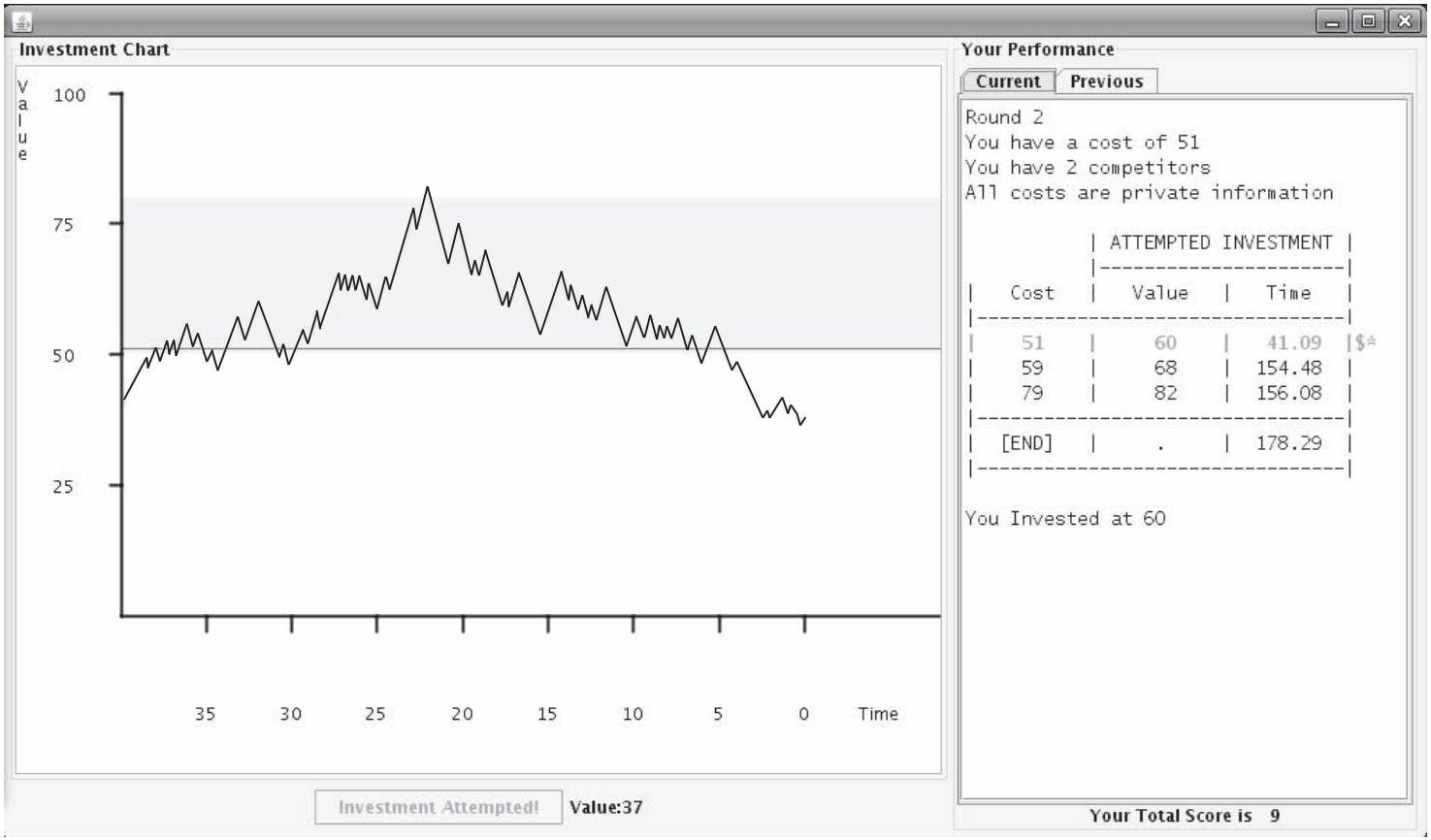
Competition groups are randomly reassigned each period.

Entry costs are independent private draws from publicly known uniform distribution on $[50, 80]$.

Participants observe same changing market value as their current rivals.

Random walk and random expiration implement same stopping environment as in single-firm experiment.

Preemption: The Experimental Interface



Anderson, Friedman, & Oprea (2010 AER), Figure 3. End-of-period screen.

Value path is common; own cost is private. Shaded band is publicly known cost range.

At the end, panel reveals rival costs and attempted entry decisions.

- Only first entrant earns a profit.

Making Losing Decisions Observable

Participants submit entry decision not knowing if rival entry had already happened.

Only first entrant earns market value minus its cost.

At the end, participants learn all costs, attempted entry times, and profits.

Records intended decisions even for firms that lose the race.

Since preempted firm cannot affect its payoff, concealing rivals' entry until the end preserves equilibrium incentives.

Random expiration still censors intended decisions that are never reached.

Analysis restricted to periods with sufficiently high realised market values.

Experimental Treatments and Predictions

Market structure: monopoly versus three contestants.

Value-process parameters: Low versus High monopoly option premium.

Predicted monopoly premia are approximately 0.5 and 0.8.

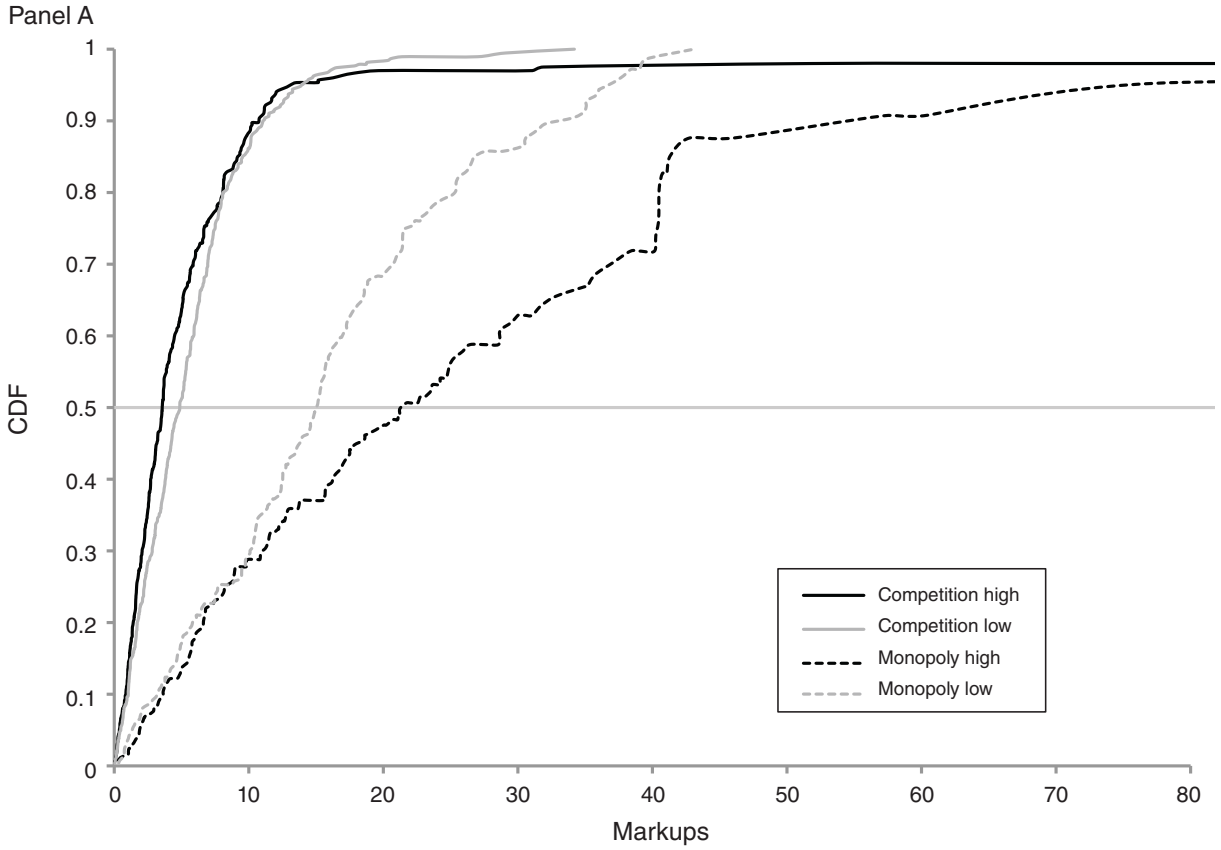
Corresponding competitive thresholds are much closer together.

Testable predictions:

1) Competition lowers entry margins.
2) Parameter treatment matters much more under monopoly.
   In line with theory – for chosen parameters, separation more significant in monopoly than competition.
3) Lowest-cost firm usually wins.

Small competitive parameter effect is prediction for these treatments.

Competition and Observed Entry Margins



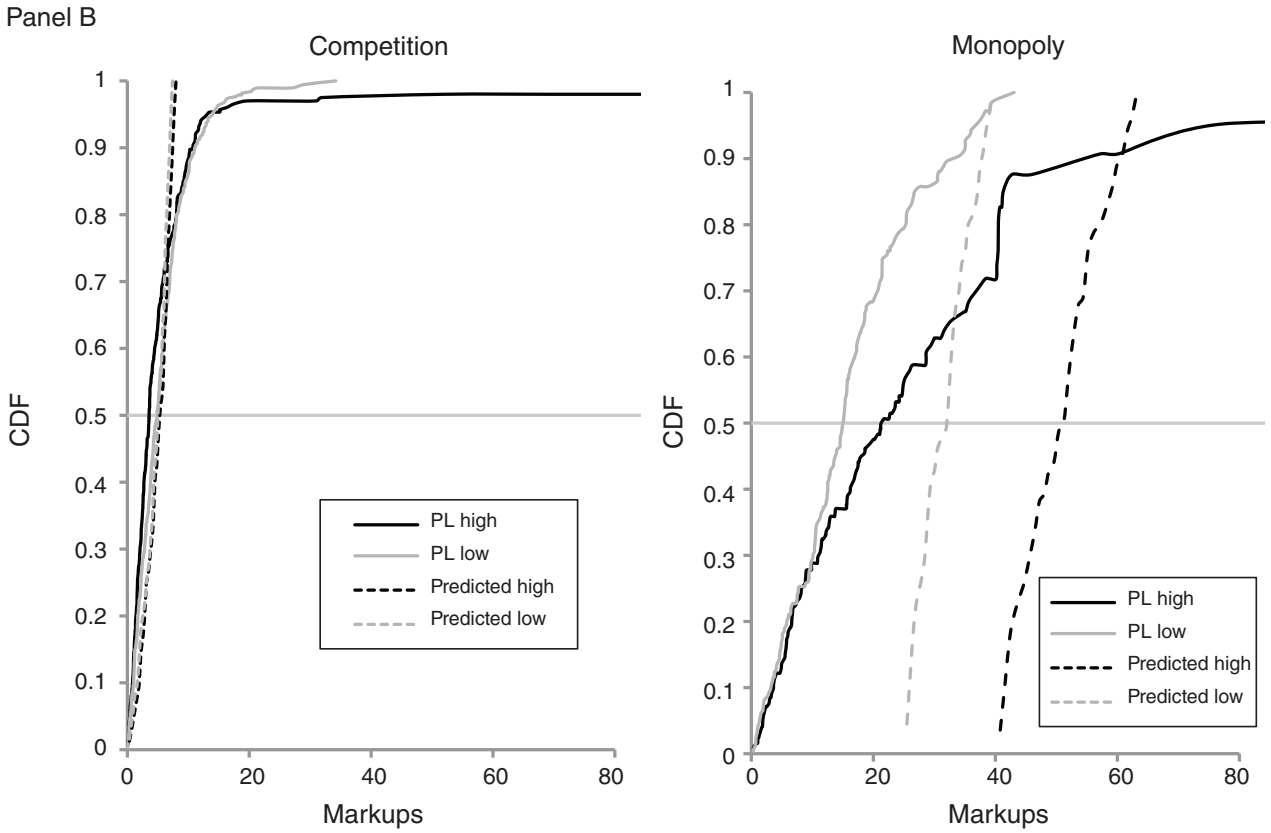
Anderson, Friedman, & Oprea (2010 AER), Figure 4A. Product-limit cdfs.

1. **Competition substantially lowers entry margins.**

2. **Parameter treatment matters much more under monopoly.**

   High and Low separate under monopoly (and ordered as per theory), but nearly coincide under competition.

Predicted and Observed Entry Margins



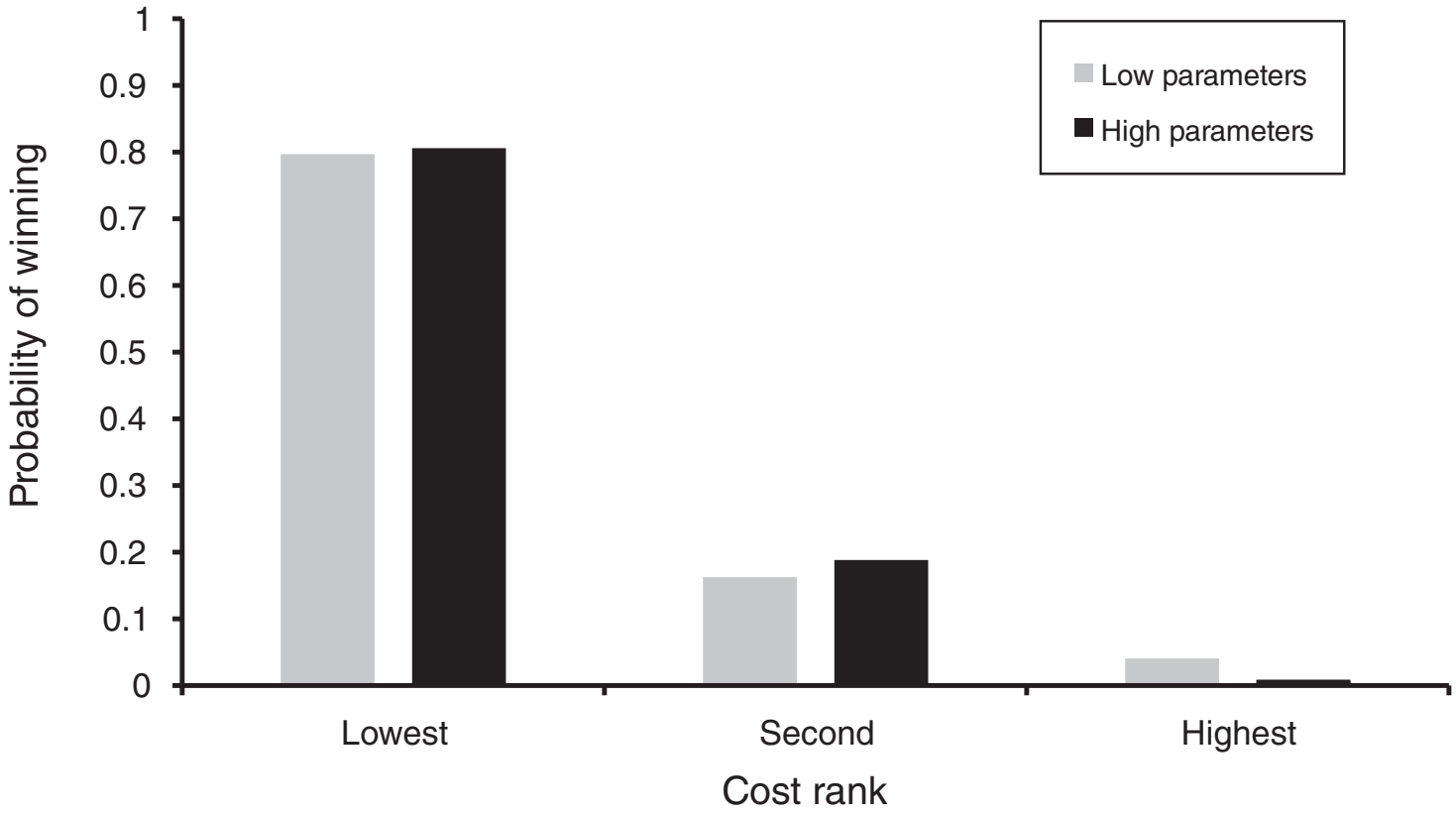
Anderson, Friedman, & Oprea (2010 AER), Figure 4B. Solid: estimated cdfs; dashed: predictions.

Competitive decisions relatively close to predictions.

Monopoly participants often choose margins below theoretical optima.

Note: CDFs are using Kaplan–Meier estimator, robust to event censoring.

Experimental Results: Who Enters?



Anderson, Friedman, & Oprea (2010 AER), Figure 5.

3. **Lowest-cost firm usually wins**; roughly 80% of the time.

When higher-cost firm wins, cost is usually close to minimum.

Supports efficient selection of entrant.

Experimental Results: Where the Theory Fits Less Well

Across full cost range, observed margins are flatter than model predicts.

- Predicted decline in margins clearer at lower costs, where expected earnings are larger.
- At high costs, chance of winning is small and mistakes in entry threshold have little effect on expected earnings.

Against observed rival behaviour, switching unilaterally to equilibrium strategy yields little additional money on average.

Poor fit to some less payoff-relevant eqm properties.

Experimental support strongest for market-structure effects and selection of entrant.

Overview

1. Entry and the Value of Waiting

2. Entry, Exit, and Hysteresis

3. Entry Contests: Preemption

4. Entry Contests: Competition and Risk Taking
   - Competing with a Privately Observed Project
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Entry Awarded to the Best Project

Seel & Strack (2013 JET), Gambling in Contests

Entry-contest interpretation: firms develop projects and decide when to submit them.

- Submission fixes project's value.
- Firm submitting highest value wins access to market.
- Project values and submission decisions are private until the contest is resolved.

Waiting can improve a project, but can also destroy its value.

Prize rewards relative performance, rather than project's expected value.

Alternative interpretations: competing for funding, promotion, or performance bonus.

The Entry Contest

$n \geq 2$ contestants privately observe independent processes

$$X_t^i = x_0 + \mu t + \sigma B_t^i, \qquad x_0 > 0, \quad \sigma > 0, \quad \mu \leq 0.$$

Each chooses possibly randomised stopping time τ_i based on own observations.

Require $\mathbb{E}[\tau_i] < \infty$ and stop by bankruptcy:

$$\tau_i \leq \inf\{t \geq 0 \mid X_t^i = 0\}.$$

Highest submitted value wins unit prize; break ties uniformly at random.

No flow payoff or discounting: $f = 0$, $r = 0$.

Unlike race to enter first, submitting early does not prevent rival from continuing.

A Rival's Strategy Determines the Scrap Value

Look for symmetric equilibrium with an atomless cdf G of submitted values.

If firm submits value x, it beats all $n-1$ rivals with probability

$$v(x; G) = G(x)^{n-1}.$$

Best-response problem is

$$V(x; G) = \sup_{\tau \in \mathbb{T}} \mathbb{E}^X[v(X_\tau; G)].$$

Optimal stopping problem, with endogenous scrap value.

Equilibrium requires best response that induces exactly cdf used to calculate v.

Need both optimality and feasibility of proposed distribution.

Equilibrium Indifference

Seek equilibrium in which firm is indifferent between stopping and continuing on $(0, \bar{x})$.

Generator and HJB. Indifference requires

$$\mathcal{L}v = \mu v'(x) + \frac{1}{2}\sigma^2 v''(x) = 0.$$

Boundary values are $v(0) = 0$ and $v(\bar{x}) = 1$.

Higher submitted value raises chance of winning.

Waiting to obtain it also raises risk of worse outcome.

Equilibrium balances these effects at every value in support.

Symmetry pins down upper endpoint through $v(x_0) = 1/n$.

Transformation with Zero Drift

Definition (Scale function)

The **scale function** of a time-homogeneous Itô process $\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$ is given by $s : \mathbb{R} \to \mathbb{R}$ satisfying $s''/s' = -2\mu/\sigma^2$. s is strictly increasing and has an inverse admitting an absolutely continuous derivative. Furthermore, s is unique up to positive affine transformations.

Definition (Speed density function)

The **speed density function** of a time-homogeneous Itô process $\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$ is given by $m : \mathbb{R} \to \mathbb{R}$ satisfying $m = 2/(\sigma^2 s')$.

Definition (Green's function)

The **Green's function** of a time-homogeneous Itô process $\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$ is given by

$$G_{a,b}(x,y) := \frac{(s(x \wedge y) - s(a))(s(b) - s(x \vee y))}{s(b) - s(a)}.$$

Transformation with Zero Drift

Theorem (Properties of scale and speed)

Let X be a time-homogeneous Itô process $\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$ and $\tau_a := \inf\{t > 0 \mid X_t = a\}$. Write $\frac{\mathrm{d}}{\mathrm{d}s} := \frac{1}{s'(x)}\frac{\mathrm{d}}{\mathrm{d}x}$ and $\frac{\mathrm{d}}{\mathrm{d}m} := \frac{1}{m'(x)}\frac{\mathrm{d}}{\mathrm{d}x}$

Then, (1) $\mathbb{P}^x(\tau_a \leq \tau_b) = \frac{s(x)-s(a)}{s(b)-s(a)}$, (2) $\mathbb{E}^x[\tau_a \wedge \tau_b] = \int_a^b G_{a,b}(x,y)m(y)\,\mathrm{d}y$, and (3) $\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}m}$.

Theorem (Dambis–Dubins–Schwarz)

Let M be a continuous local martingale adapted to $\mathcal{F}_t$ with $M_0 = 0$ and $[M]_\infty = \infty$ a.s. Let $T_u := \inf\{t > 0 \mid [M]_t > u\}$. Then, (1) $B_u := M_{T_u}$ is a standard Brownian motion with respect to $\mathcal{G}_u = \mathcal{F}_{T_u}$ and (2) $M_t = B_{[M]_t}$ for all $t \geq 0$ a.s.

Corollary

Let X be a time-homogeneous Itô process $\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$ and s its scale function. Define $Y_t := s(X_t)$. Then, $Y_t = B_{[Y]_t}$, where B is a standard Brownian motion.

Equilibrium Distribution

Scale function: Define $s(x) = \dfrac{e^{-2\mu x/\sigma^2} - 1}{-2\mu/\sigma^2}$ if $\mu < 0$ and $s(x) = x$ if $\mu = 0$.

Note: $\mathcal{L}s = 0$ and $s(0) = 0$.

First Exit Time from $(0, \bar{x})$: $\mathbb{P}^X(X_{\tau_{0,\bar{x}}} = \bar{x}) = \mathbb{P}^X(\tau_0 \leq \tau_{\bar{x}}) = s(x)/s(\bar{x})$.

Indifference between submitting now and taking this gamble gives $v(x) = s(x)/s(\bar{x})$.

Contestant wins with probability $1/n$ in symmetric atomless equilibrium:

$$v(x_0) = \frac{1}{n} \quad \Longrightarrow \quad s(\bar{x}) = n\, s(x_0).$$

Pins down upper endpoint and winning-payoff function.

Proposition (Symmetric Equilibrium Outcomes)

There is a symmetric equilibrium with submitted-value cdf

$$G(x) = \left(\frac{s(x)}{n\, s(x_0)}\right)^{1/(n-1)}, \qquad 0 \leq x \leq \bar{x}, \quad \bar{x} : s(\bar{x}) = n\, s(x_0).$$

Equilibrium Distribution

Distribution of outcomes is not a stopping strategy.

WT construct feasible strategy and show no deviation improves chance of winning.

Proposition (Symmetric Equilibrium Strategies)

With $n = 2$, independently draw $R_i \sim U(0,1)$ at time zero and stop at

$$\tau_i = \inf\left\{ t \geq 0 \,\middle|\, \frac{s(X_t^i)}{s(x_0)} \notin (1 - R_i, 1 + R_i) \right\}.$$

These strategies induce the proposed equilibrium cdf G.

Each contestant chooses interval in transformed project value, then submits on reaching either endpoint.

Allows for both good and bad outcomes.

- For $\mu < 0$, immediate submission maximises expected value.
- However, incentive to gamble due to relative-performance rule.
- Contest's allocation rule determines what a firm waits for.

Equilibrium Distribution

Proof

Conditional on $R_i = a$, stopped martingale $Y_t = s(X_t^i)/s(x_0)$ starts at 1 and ends at $1 - a$ or $1 + a$. Optional stopping gives

$$\mathbb{P}(Y_{\tau_i} = 1 - a \mid R_i = a) = \mathbb{P}(Y_{\tau_i} = 1 + a \mid R_i = a) = 1/2.$$

Mixing over $a \sim U(0, 1)$ makes Y_{τ_i} uniform on $[0, 2]$. Thus

$$\mathbb{P}(X_{\tau_i}^i \leq x) = \mathbb{P}\left(Y_{\tau_i} \leq \frac{s(x)}{s(x_0)}\right) = \frac{s(x)}{2s(x_0)}, \qquad x \leq \bar{x}.$$

Against candidate cdf, scrap value is $v(x; G) = \min\left\{\frac{s(x)}{n\, s(x_0)}, 1\right\}$.

Nonnegative scale process Y_t, absorbed at zero, is supermartingale.

$\min\{y/n, 1\}$ increasing and concave $\implies$ $v(X_t; G)$ is bounded supermartingale.

Optional stopping: $\mathbb{E}[v(X_\tau; G)] \leq v(x_0; G) = \frac{1}{n}$.

Symmetric strategy attains upper bound $1/n$ $\implies$ best response to G.

More Than Two Contestants

The target transformed outcome $Y = s(X_\tau)/s(x_0)$ has cdf

$$\mathbb{P}(Y \leq y) = \left(\frac{y}{n}\right)^{1/(n-1)}, \qquad 0 \leq y \leq n.$$

Equivalently, Y has the distribution of nU^{n-1} for $U \sim U(0,1)$.

Its mean exactly starting transformed value:

$$\mathbb{E}Y = n \int_0^1 u^{n-1}\,\mathrm{d}u = 1.$$

Similar argument goes through.

More Competition

Proposition (Comparative statics)

In the symmetric equilibrium, as n increases:

Largest submitted value $\bar{x} = s^{-1}(n\, s(x_0))$ increases.

Winning probability of each contestant decreases.

If $\mu < 0$, expected submitted value decreases and expected duration rises.

Proof idea:

$Y_n = nU^{n-1}$ becomes more dispersed with same mean.

For $\mu < 0$, s^{-1} strictly concave: extra dispersion in transformed value decreases expected submissions $\mathbb{E}[s^{-1}(s(x_0)Y_n)]$.

More rivals induce a more ambitious upper outcome and more disappointing outcomes.

Also: for $n = 2$ and $\mu < 0$, expected submissions $\mathbb{E}X_\tau$ single troughed in σ and in μ!

Two Different Effects of Competition

Risk Taking: Best submitted project wins

Rival keeps developing its project after I stop.

More rivals induce wider distribution of submissions and more risk taking.

Preemption: First entrant wins

Rival's entry destroys my opportunity.

More rivals reduce option value to waiting and bring entry forward.

Effect of competition depends on what wins the market and on what firms observe.

Overview

1. Entry and the Value of Waiting

2. Entry, Exit, and Hysteresis

3. Entry Contests: Preemption

4. Entry Contests: Competition and Risk Taking

5. Concluding Remarks

What Determines When Firms Act?

Entry: waiting preserves the right to avoid a bad commitment.

Entry and exit: changing regime destroys one option and creates another.

Best-project contest: waiting can improve relative performance even while lowering expected project value.

Preemption: waiting risks losing the opportunity altogether.

Across applications, identify the state, the flow payoff, and the scrap value first.

Use HJB and boundary conditions to find the rule; use martingales to understand what waiting buys.

Next lecture: what changes when waiting also reveals an unknown parameter?

Entry, Exit, and Strategic Entry

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Topics in Economic Theory

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Candidate Equilibrium Strategy: Existence and Uniqueness

Theorem Picard–Lindelöf

Consider the initial-value problem $y'(t) = F(t, y(t))$, $y(t_0) = y_0$. Suppose that, in a neighbourhood of (t_0, y_0), F is continuous and Lipschitz continuous in y. Then, for some $\delta > 0$, the problem has a unique continuously differentiable solution on $(t_0 - \delta, t_0 + \delta)$.

Sufficient condition: F and $\partial_y F$ continuous near (t_0, y_0).

Conclusion is *local*: it does not guarantee existence over an entire time interval.

Candidate Equilibrium Strategy: Existence and Uniqueness

Existence and Uniqueness: Let $F(l,b) := (n-1)\frac{h(l)}{1-H(l)} b \frac{b-l}{b-\beta(b-l)}$.

WTS $\exists! b : b' = F(l,b)$ on $l \in [\underline{l}, \bar{l})$ with $l < b(l) < b_M(l) = \beta/(\beta-1)l$ for $l < \bar{l}$ and $b(\bar{l}) = \bar{l}$.

Start at $l = \underline{l}$, choosing initial value $b(\underline{l}) = z \in (\underline{l}, b_M(\underline{l}))$. Picard–Lindelöf gives unique solution until $l < b(l) < b_M(l)$ is violated or reaches $\bar{l}$.

There are two possible exits before reaching $\bar{l}$:

(1) Through the lower boundary: near $b = l$, $(b-l)' = F(l,b) - 1 \to -1$. Initial values sufficiently close to lower boundary exit there.

Claim: $A := \{J \mid \text{solution starting at } b(\underline{l}) = J \text{ exits through } l \text{ before reaching } b_M \text{ or } \bar{l}\}$ is open.

Proof: Suppose $J \in A$. Since $\lim_{b \downarrow l}(b-l)' = \lim_{b \downarrow l} F(l,b) - 1 = -1$, sufficiently close to exit we must have $F(l,b) - 1 < -1/2$.

For any initial value J' sufficiently close to J, continuous dependence of the solution on starting value ensures solution stays close up to that pre-exit point. Hence, it also enters this neighbourhood and, once there, distance from lower boundary decreases at rate bounded away from zero, so it also exits through that boundary. Conclude all sufficiently nearby J' belong to A.

Candidate Equilibrium Strategy: Existence and Uniqueness

Existence and Uniqueness: Let $F(l,b) := (n-1)\frac{h(l)}{1-H(l)} b \frac{b-l}{b-\beta(b-l)}$.

WTS $\exists! b : b' = F(l,b)$ on $l \in [\underline{l}, \bar{l})$ with $l < b(l) < b_M(l) = \beta/(\beta-1)l$ for $l < \bar{l}$ and $b(\bar{l}) = \bar{l}$.

Start at $l = \underline{l}$, choosing initial value $b(\underline{l}) = J \in (\underline{l}, b_M(\underline{l}))$. Picard–Lindelöf gives unique solution until $l < b(l) < b_M(l)$ is violated or reaches $\bar{l}$.

There are two possible exits before reaching $\bar{l}$:

(1) Through the lower boundary: near $b = l$, $(b-l)' = F(l,b) - 1 \to -1$. Initial values sufficiently close to lower boundary exit there.

(2) Through the upper boundary: near $b = b_M(l)$, $(b_M - b)' = \frac{\beta}{\beta-1} - F(l,b) \to -\infty$. Initial values sufficiently close to the upper boundary exit there.

Claim: $A := \{J \mid$ solution starting at $b(\underline{l}) = J$ exits through l before reaching b_M or $\bar{l}\}$ is open.

Claim: $B = \{J \mid$ solution starting at $b(\underline{l}) = J$ exits through b_M before reaching l or $\bar{l}\}$ is open. Proof is analogous.

$\implies \exists$ initial value $b(\underline{l}) = J \in (\underline{l}, b_M(\underline{l}))$ s.t. solution reaches $\bar{l}$.

(Two disjoint open sets cannot cover a connected set.)

Candidate Equilibrium Strategy: Existence and Uniqueness

Existence and Uniqueness: Let $F(l,b) := (n-1)\frac{h(l)}{1-H(l)} b \frac{b-l}{b-\beta(b-l)}$.

WTS $\exists! b : b' = F(l,b)$ on $l \in [\underline{l}, \bar{l})$ with $l < b(l) < b_M(l) = \beta/(\beta-1)l$ for $l < \bar{l}$ and $b(\bar{l}) = \bar{l}$.

Start at $l = \underline{l}$, choosing initial value $b(\underline{l}) = z \in (\underline{l}, b_M(\underline{l}))$. Picard–Lindelöf gives unique solution until $l < b(l) < b_M(l)$ is violated or reaches $\bar{l}$.

Claim: $\exists$ initial value $b(\underline{l}) = J \in (\underline{l}, b_M(\underline{l}))$ s.t. solution reaches $\bar{l}$.

Claim: Solution b has $b(\bar{l}) = \bar{l}$.

Proof: $b' > 0$ and $b < b_M$ implies solution has finite limit $b(\bar{l}) \geq \bar{l}$.
If inequality holds strict, then near $\bar{l}$, $b'(l) \geq kh(l)/(1-H(l))$ for some $k > 0$ and then $b_M(\bar{l}) \geq b(\bar{l}) \geq b(l) + k\int_l^{\bar{l}} h(\hat{l})/(1-H(\hat{l}))\, d\hat{l} = \infty$ a contradiction.
Hence the solution does satisfy $b(\bar{l}) = \bar{l}$.

Uniqueness: Since F is strictly increasing in b, if two solutions satisfy $b(l_0) > \hat{b}(l_0)$, then $b'(l) - \hat{b}'(l) = F(l_0, b(l)) - F(l_0, \hat{b}(l)) > 0$, so the gap only increases. However, $\lim_{l\uparrow} b(l) - \hat{b}(l) = \bar{l}$, a contradiction.

Conclude: there is a unique solution $b : b' = F(l,b)$ on $l \in [\underline{l}, \bar{l})$ with $l < b(l) < b_M(l) = \beta/(\beta-1)l$ for $l < \bar{l}$ and $b(\bar{l}) = \bar{l}$.

Comparative Statics in the Preemption Game

Let

$$F(l, b; \beta, n) = (n-1)\frac{h(l)}{1-H(l)}\frac{b(b-l)}{b-\beta(b-l)}.$$

For $l < b < b_M(l)$, the denominator is positive and

$$F_\beta > 0, \qquad F_n > 0.$$

Threshold solves $b' = F$, with common terminal value $b(\bar{l}) = \bar{l}$.

Backward ODE comparison implies lower b when either β or n increases.

That $\beta_\sigma < 0$, $\beta_\mu < 0$, and $\beta_r > 0$ follows from definition.